

# Solving some Navier-Stokes Equations with the initial conditions being some complex-valued periodic functions on $\mathbb{R}^3$

Tao Zhang<sup>a</sup>, Alatancang Chen<sup>a,b,\*</sup>, Fan Bai<sup>a</sup>

<sup>a</sup>*School of Mathematical Sciences of Inner Mongolia University, Hohhot, 010021, China*

<sup>b</sup>*Huhot University for Nationalities, Hohhot, 010051, China*

---

**Abstract.** In this paper, we utilize some series and an iterative method to solve some Navier-Stokes equations with the initial conditions being some complex-valued periodic functions on  $\mathbb{R}^3$ . Then a new strategy for dealing with the conjecture of the Navier-Stokes equation is given.

**Key words:** Iterative method; Navier-Stokes equation

---

## 1 Introduction

Notation

$\mathbb{R}$  – the real numbers.

$\mathbb{C}$  – the complex numbers.

$\mathbb{R}^n = \{(r_1, \dots, r_n) \mid r_j \in \mathbb{R}, j = 1, 2, \dots, n\}$ .

$\mathbb{N}^n = \{(k_1, \dots, k_n) \mid k_j = 0, 1, 2, \dots, j = 1, 2, \dots, n\}$ .

$\mathbb{N}_+^n = \{(k_1, \dots, k_n) \mid k_j = 1, 2, \dots, j = 1, 2, \dots, n\}$ .

$\mathbb{Z}^n = \{(k_1, \dots, k_n) \mid \pm k_j \in \mathbb{N}, j = 1, 2, \dots, n\}$ .

$e_1 = (2\pi, 0, 0)$ ,  $e_2 = (0, 2\pi, 0)$ ,  $e_3 = (0, 0, 2\pi)$ .

$\varphi_k = \exp(ik_1x_1 + ik_2x_2 + ik_3x_3)$ ,  $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ .

$\wedge_{a,b,c} = \{(k_1, k_2, k_3) \mid (ak_1, bk_2, ck_3) \in \mathbb{N}^3\}$ ,  $a, b, c = \pm 1$ .

**Definition 1.1.** The set

$$\{f \in C^\infty(\mathbb{R}^3) \mid D^\beta f = \sum_{k \in \mathbb{Z}^3} h_k D^\beta \varphi_k, \beta \in \mathbb{N}^3, \{h_k\}_{k \in \mathbb{Z}^3} \subseteq \mathbb{C}\}$$

is a linear space, we use  $IE(\mathbb{R}^3)$  to denote this space.

The existence and smoothness of the Navier-Stokes equation is an open problem [1]. Consider the following question with respect to Navier-Stokes equation:

---

\*Corresponding author. Email address: zhangtaocx@163.com (T. Zhang), alatanca@imu.edu.cn (A. Chen), bf123.student@sina.com (F. Bai).

**Question 1.2.** Whether there exist  $p(x,t), u_j(x,t)$  ( $u_j(x+e_i, t) = u_j(x, t)$ ,  $i, j = 1, 2, 3$ ) such that the following Navier-Stokes equation hold?

$$\begin{cases} u_{jt} + \sum_{m=1}^3 (u_m u_{jx_m} - \nu u_{jx_m x_m}) + p_{x_j} = 0, & j=1, 2, 3, \end{cases} \quad (1.1)$$

$$\begin{cases} u_{1x_1} + u_{2x_2} + u_{3x_3} = 0, & x = (x_1, x_2, x_3) \in \mathbb{R}^3, t \geq 0, \end{cases} \quad (1.2)$$

$$\begin{cases} u_j(x, 0) = \sum_{k \in \mathbb{Z}^3} A_{jk} \varphi_k \in IE(\mathbb{R}^3), & j=1, 2, 3, \end{cases} \quad (1.3)$$

where  $\nu > 0$ ,  $u_j(x, 0), j=1, 2, 3$  are real-valued functions.

**Theorem 1.3.** Clearly we have

$$(i) \quad A_{jk} = \frac{1}{8\pi^3} \int_{[0, 2\pi]^3} u_j(x, 0) \varphi_{-k} dx_1 dx_2 dx_3, \quad k \in \mathbb{Z}^3, j=1, 2, 3.$$

$$(ii) \quad \sum_{k \in \mathbb{Z}^3} |A_{jk} k_1^{m_1} k_2^{m_2} k_3^{m_3}|^2 < \infty, \quad j=1, 2, 3, (m_1, m_2, m_3) \in \mathbb{N}^3.$$

(iii) By the conditions (1.2) and (1.3) we get

$$\sum_{k \in \mathbb{Z}^3} \sum_{j=1}^3 A_{jk} k_j \varphi_k = 0, \quad k \in \mathbb{Z}^3.$$

Note that the sequence  $\{\varphi_k\}_{k \in \mathbb{Z}^3}$  is linearly independent, so we have

$$\sum_{j=1}^3 A_{jk} k_j = 0, \quad k \in \mathbb{Z}^3.$$

$$(iv) \quad \sum_{k \in \Lambda_{a,b,c}} A_{jk} \varphi_k \in IE(\mathbb{R}^3), \quad j=1, 2, 3, a, b, c = \pm 1.$$

In this paper, base on the idea of paper [2], we can solve the following PDEs in some cases:

$$\begin{cases} \text{the PDEs (1.1) and (1.2),} \\ u_j(x, 0) = \sum_{k \in \Lambda_{a,b,c}} B_{jk} \varphi_k \in IE(\mathbb{R}^3), & j=1, 2, 3, \end{cases} \quad (1.4)$$

where  $a, b, c = \pm 1$ .

If we let

$$\sum_{k \in \mathbb{Z}^3} A_{jk} \varphi_k = \sum_{a,b,c=\pm 1} \sum_{k \in \Lambda_{a,b,c}} B_{jk} \varphi_k. \quad (1.5)$$

Then there should be some relation between the solution of the PDEs (1.1)-(1.3) and the solution of the PDEs (1.4). So a new strategy for dealing with the conjecture of the Navier-Stokes equation is given.

## 2 Main results

First we solve the following PDEs:

$$\begin{cases} \text{the PDEs (1.1) and (1.2),} \\ u_j(x,0) = \sum_{k \in \mathbb{N}^3} B_{jk} \varphi_k \in IE(\mathbb{R}^3), \quad j=1,2,3. \end{cases} \quad (2.1)$$

Suppose that the PDEs (2.1) has a solution satisfying:

$$\begin{cases} u_j(x,t) = \sum_{k \in \mathbb{N}^3} T_{jk}(t) \varphi_k, \quad j=1,2,3; \\ p(x,t) = \sum_{k \in \mathbb{N}^3} T_{4k}(t) \varphi_k. \end{cases} \quad (2.2)$$

Assume that the series (2.2) satisfies the following conditions:

$$\begin{cases} u_j = \sum_{k \in \mathbb{N}^3} T_{jk}(t) \varphi_k \in C(\mathbb{R}^3 \oplus [0, +\infty)), \quad j=1,2,3, \end{cases} \quad (2.3)$$

$$p = \sum_{k \in \mathbb{N}^3} T_{4k}(t) \varphi_k \in C(\mathbb{R}^3 \oplus [0, +\infty)), \quad (2.4)$$

$$u_{jt} = \sum_{k \in \mathbb{N}^3} T'_{jk}(t) \varphi_k \in C(\mathbb{R}^3 \oplus [0, +\infty)), \quad j=1,2,3, \quad (2.5)$$

$$u_{jx_m} = \sum_{k \in \mathbb{N}^3} ik_m T_{jk}(t) \varphi_k \in C(\mathbb{R}^3 \oplus [0, +\infty)), \quad m,j=1,2,3, \quad (2.6)$$

$$u_{jx_mx_m} = \sum_{k \in \mathbb{N}^3} -k_m^2 T_{jk}(t) \varphi_k \in C(\mathbb{R}^3 \oplus [0, +\infty)), \quad m,j=1,2,3, \quad (2.7)$$

$$p_{x_j} = \sum_{k \in \mathbb{N}^3} ik_j T_{4k}(t) \varphi_k \in C(\mathbb{R}^3 \oplus [0, +\infty)), \quad j=1,2,3, \quad (2.8)$$

$$u_m u_{jx_m} = \sum_{k \in \mathbb{N}^3} \eta_{mjk} \varphi_k \in C(\mathbb{R}^3 \oplus [0, +\infty)), \quad m,j=1,2,3, \quad (2.9)$$

where

$$\eta_{mjk} = \sum_{k^{[1]}+k^{[2]}=k} k_m^{[2]} T_{mk^{[1]}} T_{jk^{[2]}}, \quad k^{[l]} = (k_1^{[l]}, k_2^{[l]}, k_3^{[l]}) \in \mathbb{N}^3, \quad l=1,2, \quad m,j=1,2,3.$$

Then substituting the series (2.2) into the equations (2.1) we get

$$\begin{cases} T'_{m,(0,0,0)} + \sum_{k > (0,0,0)} [T'_{mk} + \sum_{j=1}^3 (\sum_{k^{[1]}+k^{[2]}=k} ik_j^{[2]} T_{jk^{[1]}} T_{mk^{[2]}} + \nu k_j^2 T_{mk}) + ik_m T_{4k}] \varphi_k = 0, \quad m=1,2,3, \\ \sum_{k \in \mathbb{N}^3} (ik_1 T_{1k} + ik_2 T_{2k} + ik_3 T_{3k}) \varphi_k = 0, \\ u_j(x,0) = \sum_{k \in \mathbb{N}^3} B_{jk} \varphi_k = \sum_{k \in \mathbb{N}^3} T_{jk}(0) \varphi_k, \quad j=1,2,3. \end{cases}$$

Note that the sequence  $\{\varphi_k\}_{k \in \mathbb{N}^3}$  is linearly independent, so the above equations are equivalent to the following ODEs:

$$\begin{cases} T'_{j,(0,0,0)} = 0, \\ T_{j,(0,0,0)}(0) = B_{j,(0,0,0)}, \end{cases} \quad j=1,2,3,$$

and

$$\left\{ \begin{array}{l} T'_{mk} + \sum_{j=1}^3 \left( \sum_{k^{[1]}+k^{[2]}=k} ik_j^{[2]} T_{jk^{[1]}} T_{mk^{[2]}} + \nu k_j^2 T_{mk} \right) + ik_m T_{4k} = 0, \quad m=1,2,3, \\ k_1 T_{1k} + k_2 T_{2k} + k_3 T_{3k} = 0, \\ T_{jk}(0) = B_{jk}, \quad j=1,2,3, \end{array} \right.$$

where  $k > (0,0,0)$ . By the equations  $k_1 T_{1k} + k_2 T_{2k} + k_3 T_{3k} = 0$ ,  $k > (0,0,0)$ , we have

$$k_1 T'_{1k} + k_2 T'_{2k} + k_3 T'_{3k} = 0, \quad k > (0,0,0).$$

Then we obtain

$$\sum_{m=1}^3 k_m \sum_{j=1}^3 \sum_{\substack{k^{[1]}+k^{[2]}=k, \\ k^{[1]}, k^{[2]} > (0,0,0)}} ik_j^{[2]} T_{jk^{[1]}} T_{mk^{[2]}} + T_{4k} \sum_{m=1}^3 ik_m^2 = 0, \quad k > (0,0,0).$$

So we get

$$\left\{ \begin{array}{l} T_{j,(0,0,0)}(t) = B_{j,(0,0,0)}, \quad j=1,2,3, \\ T_{4,(0,0,0)} = a, \quad a \text{ is an arbitrary constant,} \\ T_{4k}(t) = \frac{-\sum_{m=1}^3 k_m \sum_{j=1}^3 \sum_{\substack{k^{[1]}+k^{[2]}=k, \\ k^{[1]}, k^{[2]} > (0,0,0)}} k_j^{[2]} T_{jk^{[1]}} T_{mk^{[2]}}}{\sum_{m=1}^3 k_m^2}, \quad k > (0,0,0), \\ T_{jk}(t) = \exp(-P_k(t)) \left( \int_0^t Q_{jk}(s) \exp(P_k(s)) ds + B_{jk} \right), \quad j=1,2,3, \quad k > (0,0,0), \end{array} \right.$$

where

$$\left\{ \begin{array}{l} Q_{mk} = -\sum_{j=1}^3 \sum_{\substack{k^{[1]}+k^{[2]}=k, \\ k^{[1]}, k^{[2]} > (0,0,0)}} ik_j^{[2]} T_{jk^{[1]}} T_{mk^{[2]}} - ik_m T_{4k}, \quad m=1,2,3, \\ P_k(t) = \sum_{j=1}^3 ik_j B_{j,(0,0,0)} t + \nu \sum_{j=1}^3 k_j^2 t. \end{array} \right.$$

Clearly we have:

**Theorem 2.1.** If the series (2.2) we obtain satisfies the conditions (2.3)-(2.9), then it is a solution of the PDEs (2.1).

By the Abel identities [3] we have:

**Lemma 2.2.** For any  $k=1,2,\dots$ , we have

- (i)  $\sum_{m=1}^k \frac{(k+1)!}{m!(k+1-m)!} m^m (k+1-m)^{k-m} = k(k+1)^k,$
- (ii)  $\sum_{m=1}^k \frac{(k+1)!}{m!(k+1-m)!} m^{m-1} (k+1-m)^{k-m} = 2k(k+1)^{k-1} \leq 2(k+1)^k.$

**Corollary 2.3.** For any  $k = (k_1, \dots, k_n) \in \mathbb{N}_+^n$ , we have

$$\sum_{(1, \dots, 1) \leq (m_1, \dots, m_n) \leq k} m_1 \prod_{j=1}^n \frac{m_j^{m_j-1} (k_j+1-m_j)^{k_j-m_j}}{m_j! (k_j+1-m_j)!} \leq 2^{n-1} k_1 \prod_{j=1}^n \frac{(k_j+1)^{k_j}}{(k_j+1)!}.$$

**Lemma 2.4.** Let  $k = (k_1, k_2, k_3) \in \mathbb{N}^3$ ,  $|k| = |k_1 + k_2 + k_3|$ ,  $\nu \geq 1$ . If

$$|B_{jk}| \leq \frac{e^{-|k|}}{10^3} \prod_{j=1,2,3, k_j > 0} \frac{k_j^{k_j-1}}{k_j!}, \quad k > (0,0,0), j=1,2,3.$$

Then we have

$$|T_{ik}(t)| \leq \frac{1}{100} \prod_{j=1,2,3, k_j > 0} \frac{k_j^{k_j-1}}{k_j!} \exp(-\nu|k|t - |k|), \quad k > (0,0,0), i=1,2,3. \quad (2.10)$$

**Proof.** We prove the inequalities (2.10) by the induction method. By a simple calculate we can induce that the inequalities (2.10) hold when  $|k| = 1, 2$ . Suppose that it hold for any  $|k| < n$  ( $n > 2$ ), then by Corollary 2.3, for any  $k = (k_1, k_2, k_3) \geq (1, 1, 1)$ ,  $|k| = n$  (without loss of generality we suppose that  $k_1 \geq k_2, k_3$ ), we have

$$\begin{aligned} |T_{4k}(t)| &\leq \frac{\sum_{m=1}^3 k_m \sum_{j=1}^3 \sum_{k^{[1]}+k^{[2]}=k, k^{[1]}, k^{[2]} > (0,0,0)} k_j^{[2]} |T_{jk^{[1]}}| |T_{mk^{[2]}}|}{\sum_{m=1}^3 k_m^2} \\ &\leq \frac{3 \sum_{m=1}^3 k_m}{10^4 \sum_{m=1}^3 k_m^2} \left( \sum_{1 \leq m_i \leq k_i, i=1,2,3} m_1 \prod_{j=1}^3 \frac{m_j^{m_j-1} (k_j - m_j)^{k_j - m_j - 1}}{m_j! (k_j - m_j)!} + 6k_1 \prod_{j=1}^3 \frac{k_j^{k_j-1}}{k_j!} \right) \exp(-\nu|k|t - |k|) \\ &\leq \frac{60}{10^4} \prod_{j=1}^3 \frac{k_j^{k_j-1}}{k_j!} \exp(-\nu|k|t - |k|), \\ |Q_{jk}(s)| &\leq \sum_{j=1}^3 \sum_{k^{[1]}+k^{[2]}=k, k^{[1]}, k^{[2]} > (0,0,0)} k_j^{[2]} |T_{jk^{[1]}}(s)| |T_{mk^{[2]}}(s)| + k_j |T_{4k}(s)| \\ &\leq \frac{90k_1}{10^4} \prod_{j=1}^3 \frac{k_j^{k_j-1}}{k_j!} \exp(-\nu|k|t - |k|), \quad j=1,2,3, \\ |T_{jk}(t)| &\leq \exp(-\nu \sum_{j=1}^3 k_j^2 t) \left( \int_0^t |Q_{jk}(s)| \exp(\nu \sum_{j=1}^3 k_j^2 s) ds + |B_{jk}| \right) \\ &\leq \frac{1}{100} \prod_{j=1}^3 \frac{k_j^{k_j-1}}{k_j!} \exp(-\nu|k|t - |k|), \quad j=1,2,3. \end{aligned}$$

In a similar way, we can prove that the inequalities (2.10) hold for any  $k = (k_1, k_2, k_3) \in \mathbb{N}^3$  with  $k_1 = 0$  or  $k_2 = 0$  or  $k_3 = 0$ .

**Theorem 2.5.** Let  $|k| = |k_1 + k_2 + k_3|$ ,  $\nu \geq 1$ . If

$$|B_{jk}| \leq \frac{e^{-|k|}}{10^3} \prod_{j=1,2,3, k_j > 0} \frac{k_j^{k_j-1}}{k_j!}, \quad k > (0,0,0), j=1,2,3.$$

Then the series (2.2) we obtain is a solution of the PDEs (2.1).

**Proof.** We only need to prove that the series (2.2) we obtain satisfies the conditions (2.3)- (2.9). Note that  $\frac{k^m}{m!} \leq e^k$ ,  $m \in \mathbb{N}$ , so we have

$$\begin{cases} |T_{ik}(t) \varphi_k| \leq \frac{1}{100} e^{-\nu|k|t}, & k > (0,0,0), i=1,2,3, \\ |T_{4k}(t) \varphi_k| \leq \frac{60}{10^4} e^{-\nu|k|t}, & k > (0,0,0). \end{cases}$$

Hence the series (2.2) converges absolutely on  $\Omega \oplus [0, +\infty)$ . It means that the series (2.2) we obtain satisfies the conditions (2.3)-(2.4). Moreover, we can prove that

$$\left\{ \begin{array}{l} |T'_{mk}\varphi_k| = \left| \sum_{j=1}^3 \sum_{k^{[1]}+k^{[2]}=k} ik_j^{[2]} T_{jk^{[1]}} T_{mk^{[2]}} + \nu k_j^2 T_{mk} \right| + |ik_m T_{4k}| |\varphi_k| \\ \leq (\nu |k|^2 |T_{mk}| + |Q_{mk}|) < \left( \frac{\nu |k|^2}{100} + \frac{90|k|}{10^4} \right) e^{-\nu |k|t}, \quad m=1,2,3, k > (0,0,0), \\ |k_j^2 T_{mk}\varphi_k| \leq \frac{|k|^2}{100} e^{-\nu |k|t}, \quad m,j=1,2,3, k > (0,0,0), \\ |k_j T_{4k}\varphi_k| \leq \frac{60|k|}{10^4} e^{-\nu |k|t}, \quad j=1,2,3, k > (0,0,0), \\ |\eta_{jmk}\varphi_k| \leq \frac{10}{10^4} e^{-\nu |k|t}, \quad m,j=1,2,3, k > (0,0,0). \end{array} \right.$$

So the series (2.2) we obtain satisfies the conditions (2.5)-(2.9). Therefore it is a solution of the equations (2.1) by Theorem 2.1.

Similarly, we can get:

**Theorem 2.6.** Let  $\nu \geq 1$ . For any  $a,b,c = \pm 1$ , if

$$|B_{jk}| \leq \frac{\exp(-\sum_{j=1}^3 |k_j|)}{10^3} \prod_{j=1,2,3, |k_j|>0} \frac{|k_j|^{|k_j|-1}}{|k_j|!}, \quad (ak_1, bk_2, ck_3) > 0, j=1,2,3,$$

then the solution of the PDEs (1.4) exists.

**Theorem 2.7.** For any  $a,b,c = \pm 1$ , if the functions  $p, u_j, j=1,2,3$  satisfy the PDEs (1.4), then  $\bar{p}, \bar{u}_j, j=1,2,3$  satisfy the following PDEs

$$\left\{ \begin{array}{l} \text{the PDEs (1.1) and (1.2),} \\ u_j(x,0) = \sum_{k \in \Lambda_{a,b,c}} \bar{B}_{jk} \bar{\varphi}_k, \quad j=1,2,3. \end{array} \right.$$

Next suppose that the equality (1.5) holds, then we give the following conjectures:

**Conjecture 2.8.** If for any  $a,b,c = \pm 1$ , the solution of the PDEs (1.4) exists (and unique), then the solution of the PDEs (1.1)-(1.3) exists (and unique).

**Conjecture 2.9.** Suppose that for any  $a,b,c = \pm 1$ , the functions  $p_{abc}, u_{jabc}, j=1,2,3$  satisfy the PDEs (1.4), and that the functions  $p, u_j, j=1,2,3$  satisfy the PDEs (1.1)-(1.3), then there exist some nonlinear functions  $T_j, j=1,2,3,4$ , such that

$$\left\{ \begin{array}{l} T_j(\Delta) = u_j(x,t), \quad j=1,2,3, \\ T_4(\Delta) = p(x,t), \end{array} \right.$$

where  $\Delta = \{p_{abc}(x,t), u_{jabc}(x,t) \mid j=1,2,3, a,b,c = \pm 1\}$ .

## Acknowledgments

The paper is supported by the Natural Science Foundation of China (no. 11371185) and the Natural Science Foundation of Inner Mongolia, China (no. 2013ZD01).

## References

- [1] C.L. Fefferman, Existence and smoothness of the Navier-Stokes equation. 2000.
- [2] T. Zhang, A. Chen, A new method of solving PDEs, Preprint available at arXiv:1503.07582, 2015.
- [3] J. Riordan, Combinatorial Identities, 1968.